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Homework No. 13
Finite Elements, Winter 2018/19

Problem 13.1: Error propagation operators for iterative methods

For $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$, consider solving the linear system

$$A\boldsymbol{x} = \boldsymbol{b}$$

by some iterative method, which, for $G \in \mathbb{R}^{n\times n}$ and $\boldsymbol{d} \in \mathbb{R}^n$, at step $k+1$ can be written as

$$\boldsymbol{x}^{(k+1)} = G\boldsymbol{x}^{(k)} + \boldsymbol{d}.$$

(a) Let $\boldsymbol{e}^{(k)} = \boldsymbol{x} - \boldsymbol{x}^{(k)}$ denote the error at the kth step. Show that the matrix G is as also the *error propagation operator* $\mathbf{E}$, that is, show that the error at step $k+1$ can be written

$$\boldsymbol{e}^{(k+1)} = G\boldsymbol{e}^{(k)} = G^k\boldsymbol{e}^{(0)}.$$

(b) The componentwise form of the Jacobi method is given by the iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j\neq i} a_{ij}x_j^{(k)}\right), \qquad \text{for } i = 1, 2, \ldots, n.$$

Show that the error propagation operator is given by $\mathbf{E} = I - D^{-1}A$, where I is the identity and D is a diagonal matrix.
Hint: One can decompose the matrix A into $A = D + L + U$, where D is a diagonal matrix, L is lower triangular and U is upper triangular.

(c) The componentwise form of the Gauss–Seidel method is given by the iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right), \qquad \text{for } i = 1, 2, \ldots, n.$$

Derive a matrix representation of the error propagation operator $\mathbf{E}$, with a similar form as in **(b)**.

Problem 13.2: Mesh refinement and iteration count

(a) Fix an error tolerance $\varepsilon > 0$. Show that for Richardson iteration, mesh refinement by a factor of 2, $h \mapsto h/2$, increases the number of iterations required to reach an error of ε by a factor of 4.

Hint 1: Recall from Theorem 4.1.3 that for Richardson iteration the error at the kth step satisfies

$$\|\boldsymbol{e}^{(k)}\| \le \left(\frac{\kappa-1}{\kappa+1}\right)^k \|\boldsymbol{e}^{(0)}\|,$$

where κ is the spectral condition number of $\boldsymbol{A}_h$, and

$$\frac{\kappa-1}{\kappa+1} = 1 - \frac{2}{\kappa} + \mathcal{O}(\kappa^{-2}).$$

Hint 2: The Taylor series expansion of $\log(1-x)$ may be useful.

(b) Fix an error tolerance $\varepsilon > 0$. Show that for the conjugate gradient (CG) method, mesh refinement by a factor of 2, $h \mapsto h/2$, results in a doubling of the number of iterations required to reach an error of ε.

Hint: Recall from Theorem 4.2.9 that for CG the error at the kth step satisfies

$$\|\boldsymbol{e}^{(k)}\| \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|\boldsymbol{e}^{0}\|,$$

where κ is the spectral condition number of $\boldsymbol{A}_h$.