IWR – Universität Heidelberg
Prof. Dr. Guido Kanschat
Alec Gilbert

Due date: **18.01.2019**

Homework No. 12
Finite Elements, Winter 2018/19

Problem 12.1: Condition number of FE matrices

Let $\Omega \subset \mathbb{R}^d$ be a convex, polygonal domain, and let $\alpha, \beta \in C^\infty(\overline{\Omega})$. Also, assume there exists positive constants $\alpha_{\min}$, $\alpha_{\max}$, $\beta_{\min}$, $\beta_{\max}$ such that

$$\alpha_{\min} \le \alpha(\boldsymbol{x}) \le \alpha_{\max} \quad \text{and} \quad \beta_{\min} \le \beta(\boldsymbol{x}) \le \beta_{\max}, \quad \text{for all } \boldsymbol{x} \in \Omega\,.$$

Let $V = H_0^1(\Omega)$ and consider the boundary value problem

$$\begin{aligned} -\nabla \cdot (\alpha(\boldsymbol{x}) \nabla u(\boldsymbol{x})) &= \beta(\boldsymbol{x}) f(\boldsymbol{x}), && \text{for } \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) &= 0, && \text{for } \boldsymbol{x} \in \partial\Omega. \end{aligned} \tag{12.1}$$

Also, let $\{\mathbb{T}_h\}$ be a quasi-uniform family of meshes, and let $\{\varphi_{h,i}\}$ be the basis of a finite element shape function space for the mesh $\mathbb{T}_h$.

(a) Write out the elements of the stiffness matrix $\boldsymbol{A}_h$ and the mass matrix $\boldsymbol{M}_h$ in terms of the basis functions $\varphi_{h,i}$, corresponding to the FE discretisation of the problem (12.1).

(b) Similar to Lemma 4.3.4, derive a bound on the condition number of the mass matrix for the problem (12.1).

(c) Let $a(\cdot,\cdot)$ be a coercive, symmetric, bilinear form on V, and let $b(\cdot,\cdot)$ be an inner product on V, which for all $v \in V$ satisfy

$$\begin{aligned} a_{\min}(\nabla v, \nabla v) &\le a(v,v) \le a_{\max}(\nabla v, \nabla v) \\ b_{\min}(v,v) &\le b(v,v) \le b_{\max}(v,v), \end{aligned}$$

where $a_{\min}$, $b_{\min}$, $a_{\max}$, $b_{\max}$ are positive constants and $(\cdot,\cdot)$ denotes the L^2 inner product.

Now, consider the variational eigenvalue problem: find $\chi \in \mathbb{R}$, $u \in V$ such that

$$a(u,v) = \chi b(u,v) \quad \text{for all } v \in V. \tag{12.2}$$

Show that the smallest eigenvalue χ of (12.2) satisfies

$$\frac{a_{\min}}{b_{\max}} \mu \le \chi \le \frac{a_{\max}}{b_{\min}} \mu\,,$$

where μ is the smallest eigenvalue of the Laplacian with Dirichlet boundary conditions (see equation 4.35 in the notes). You may assume that χ and μ are simple eigenvalues.

Hint: The smallest eigenvalue of a variational eigenvalue problem can be represented by its Rayleigh quotient

$$\chi = \min_{u \in V} \frac{a(u,u)}{b(u,u)}\,.$$

(d) Similar to Theorem 4.3.7, derive a bound on the condition number of the stiffness matrix for the problem (12.1).

(e) Is the condition that the coefficients are smooth necessary?

(f) Discuss the differences to the results for the Poisson equation from the notes.