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Due date: **21.12.2018**

Homework No. 10
Finite Elements, Winter 2018/19

Problem 10.1: A posteriori error analysis for general smooth, symmetric coefficients cont.

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex, polyhedral domain and let $V = H_0^1(\Omega)$. Let $a_{i,j} \in C^\infty(\overline{\Omega})$ for $i, j = 1, 2, \dots, d$ be symmetric coefficients ($a_{i,j} = a_{j,i}$) such that for all $\boldsymbol{x} \in \Omega$ the coefficient matrix $A(\boldsymbol{x}) = [a_{i,j}(\boldsymbol{x})]$ satisfies

$$\boldsymbol{\xi}^T A(\boldsymbol{x})\boldsymbol{\xi} \ \geq \ C_0|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d\,.$$

Consider the bilinear form $a : V \times V \to \mathbb{R}$ given by

$$a(u, v) \ = \ \sum_{i,j=1}^{d} \int_\Omega a_{i,j}(\boldsymbol{x})\partial_i u(\boldsymbol{x})\partial_j v(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x}\,,$$

and define the corresponding *energy norm* on V by

$$\|u\|_V \ = \ \|u\|_a \ \coloneqq \ \sqrt{a(u, u)}\,.$$

For $f \in V^*$, the variational equation corresponding to the above bilinear form is to find $u \in V$ such that

$$a(u, v) = f(v) \quad \text{for all } v \in V. \tag{10.1}$$

Further, let $\{V_h\}_{h>0}$ be a family of conforming finite element spaces each defined on a shape regular mesh $\mathbb{T}_h$, and let $\mathbb{P}_k(T) \subset V_h|_T$ for all cells $T \in \mathbb{T}_h$.

(a) Let also $\mathbb{T}_h$ be locally quasi-uniform. Formulate and then prove an estimate of the form (2.89) from Lemma 2.3.12 in the notes.

Hint: Recall from the previous class that for $v, w \in V$ the strong form of the residual is

$$a(Rv, w) \ = \ \sum_{T \in \mathbb{T}_h} \int_T r_T(v) w \,\mathrm{d}\boldsymbol{x} - \sum_{F \in \mathbb{F}_h^i} \int_F 2\{\{\boldsymbol{n} \cdot A\nabla v\}\} w \,\mathrm{d}s\,, \tag{10.2}$$

where $r_T(v) \ \coloneqq \ f + \nabla \cdot (A\nabla v)$, and we proved that for $v \in V$ the weak residual satisfies

$$\|u - v\|_a \ = \ \|Rv\|_{-a} \ = \ \sup_{\|w\|_a = 1} |a(Rv, w)|\,. \tag{10.3}$$

(b) As in Definition 2.3.15, define the residual based error estimator $\eta_{a,h}$ for (10.1).

(c) Formulate and then prove an estimate of the form (2.95) from the notes for (10.1) and the energy norm.

(d) Formulate and then prove an estimate of the form (2.96) from the notes for (10.1) and the energy norm.