IWR – Universität Heidelberg
Prof. Dr. Guido Kanschat
Alec Gilbert

Due date: **14.12.2018**

Homework No. 9
Finite Elements, Winter 2018/19

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex, polyhedral domain and let $V = H_0^1(\Omega)$. Let $a_{i,j} \in C^\infty(\Omega)$ for $i, j = 1, 2, \dots, d$ be symmetric coefficients ($a_{i,j} = a_{j,i}$) such that for all $\boldsymbol{x} \in \Omega$ the coefficient matrix $A(\boldsymbol{x}) = [a_{i,j}(\boldsymbol{x})]$ satisfies

$$\boldsymbol{\xi}^T A(\boldsymbol{x})\boldsymbol{\xi} \ \geq\ C_0|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d\,.$$

Consider the bilinear form $a : V \times V \to \mathbb{R}$ given by

$$a(u, v) \ =\ \sum_{i,j=1}^{d} \int_\Omega a_{i,j}(\boldsymbol{x})\partial_i u(\boldsymbol{x})\partial_j v(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x}\,,$$

and define the corresponding *energy norm* on V by

$$\|u\|_V \ =\ \|u\|_a \ \coloneqq\ \sqrt{a(u, u)}\,.$$

Problem 9.1: A priori error analysis for general smooth, symmetric coefficients

(a) Show that $a(\cdot, \cdot)$ defines an inner product on V, and hence that $\|\cdot\|_a$ defined above is in fact a norm.

(b) Argue that for all $f \in V^*$ there exist a unique solution $u \in V$ to the variational problem

$$a(u, v) \ =\ f(v) \quad \text{for all } v \in V\,,$$

and that the solution satisfies

$$\|u\|_a \ =\ \|f\|_{-a}\,,$$

where the dual norm is given by

$$\|f\|_{-a} \ =\ \|f\|_{V^*} \ \coloneqq\ \sup_{\|v\|_a = 1} |a(f, v)|\,.$$

(c) Let $\{V_h\}_{h>0}$ be a family of conforming finite element spaces each defined on a shape regular mesh $\mathbb{T}_h$, and let $\mathbb{P}_k(T) \subset V_h|_T$ for all cells $T \in \mathbb{T}_h$. The discrete problem is to find $u_h \in V_h$ such that

$$a(u_h, v_h) \ =\ f(v_h) \quad \text{for all } v_h \in V_h\,.$$

Prove that the FE solution $u_h \in V_h$ satisfies the *best approximation property* with respect to the energy norm:

$$\|u - u_h\|_a \ =\ \inf_{v \in V_h} \|u - v_h\|_a\,.$$

Note that in comparison to (2.24), the constant here is 1.

(d) Prove that the energy norm is equivalent to the usual Sobolev norm $\|\cdot\|_{1,\Omega}$:

$$C_1\,\|v\|_{1,\Omega} \ \leq\ \|v\|_a \ \leq\ C_2\,\|v\|_{1,\Omega} \quad \text{for all } v \in V\,.$$

(e) Assuming additionally that $u \in H^{k+1}(\Omega)$, prove the following bound on the error of the FE solution

$$\|u - u_h\|_a \ \leq\ C_3 h^k |u|_{k+1,\Omega}\,.$$

(f) Discuss the differences to Poisson's equation. Discuss what happens if the coefficients are no longer symmetric.

Problem 9.2: A posteriori error analysis for general smooth symmetric coefficients

(a) Write the residual for $a(w, v) = f(v)$ in weak (Definition 2.3.8) and strong (Definition 2.3.11) form. Compare to the Laplacian.

(b) Prove an isometry as in Lemma 2.3.9, you will need to define suitable norms.