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Homework No. 8
Finite Elements, Winter 2018/19

Problem 8.1: Higher-order variational problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex, polyhedral domain. Let $V = H_0^2(\Omega)$ and for $f \in V^*$ (the dual space of V) consider the following variational problem: Find $u \in V$ such that

$$a(u, v) = f(v) \quad \text{for all } v \in V\,, \tag{8.1}$$

where the bilinear form $a(\cdot,\cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$ is given by

$$a(u,v) := \int_\Omega \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} u(\boldsymbol{x}) \frac{\partial^2}{\partial x_j \partial x_i} v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}\,.$$

(a) Show that any $u \in V$ also satisfies

$$\frac{\partial u}{\partial x_j} \in H_0^1(\Omega) \quad \text{for } j = 1, 2, \ldots, d\,.$$

Recall that $V = H_0^2(\Omega)$ is the completion of $C_{00}^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{2,\Omega}$.

(b) Show that

$$a(u,u) = |u|_{2,\Omega}^2 \quad \text{for all } u \in V\,.$$

(c) Prove that there exists a constant $0 < C_0 < \infty$ such that for all $u \in V$

$$\|u\|_{2,\Omega} \le C_0 |u|_{2,\Omega}\,.$$

(d) Show that for each $f \in V^*$ there exists a unique solution $u \in V$ to (8.1).

(e) For a family of shape-regular meshes $\{\mathbb{T}_h\}_{h>0}$ of Ω, consider the conforming finite element spaces $V_h \subset V$, where for each cell $T \in \mathbb{T}_h$ the restriction of V_h to T contains $P_k(T)$ for $k \ge 2$. The discrete problem is to find $u_h \in V_h$ such that

$$a(u_h, v_h) = f(v_h) \quad \text{for all } v_h \in V_h\,.$$

Assuming additionally that $u \in H^{k+1}(\Omega)$, prove the following energy error estimate

$$\|u - u_h\|_{2,\Omega} \le C_1 h^{k-1} |u|_{k+1,\Omega}\,.$$

(f) Prove the following error estimate in the H^1 norm

$$\|u - u_h\|_{1,\Omega} \le C_2 h^k |u|_{k+1,\Omega}\,,$$

and state the extra regularity assumptions that are required in order for this to hold.

Hint: Consider an appropriate dual problem and use the Aubin–Nitsche trick.

(g) Discuss the construction of such a finite element space V_h. What conditions must the functions in V_h satisfy in order to be conforming?