

Homework No. 7
Finite Elements, Winter 2018/19

Problem 7.1: H^1 - and L^2 -Error Estimates

Consider the following problem

$$\begin{aligned} -\Delta u + u &= f, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

where Ω is a convex polygonal domain in \mathbb{R}^2 .

We will use a conforming FE method with piecewise linear finite elements ($V_h \subset V$) to approximate the above problem.

- (a) Formulate the variational equation and its approximation, and name the solution space V as well as the ansatz spaces V_h .
- (b) Discuss unique existence of the exact solution $u \in V$ as well as of the discrete solutions $u_h \in V_h$.
- (c) Derive the energy error estimate

$$\|\nabla(u - u_h)\|_{L^2} \leq ch\|f\|_{L^2},$$

by assuming u has elliptic regularity.

- (d) Derive the L^2 -error estimate

$$\|u - u_h\|_{L^2} \leq ch^2\|f\|_{L^2}$$

by Aubin's and Nitsche's duality argument.

Hint: For a given pde in weak form "Find $u \in V$, s.t. $a(u, \varphi) = (f, \varphi) \forall \varphi \in V$ " the dual problem is given by "Find $z \in V$, s.t. $a(v, z) = J(v) \forall v \in V$ ", where the error functional $J(\cdot)$ can be defined appropriately to the specific target.