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Homework No. 6
Finite Elements, Winter 2018/19

Problem 6.1: Trace inequality for polynomials

Let K be a square of diameter h and ∂K its boundary. Show that for any polynomial $p \in P_k$ it holds that

$$\|p\|_{L^2(\partial K)}^2 \le C\left(h\|\nabla p\|_{L^2(K)}^2 + \frac{1}{h}\|p\|_{L^2(K)}^2\right)$$

with a constant $C > 0$.

Hint: The "scaling" lemma (Lemma 2.1.41 in the notes) may be useful.

Problem 6.2: Bramble–Hilbert for bilinear forms

The Bramble–Hilbert Lemma can be extended to bilinear forms. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, and let $a : H^{k+1}(\Omega) \times H^{\ell+1}(\Omega) \to \mathbb{R}$ be a continuous bilinear form that satisfies

$$\begin{aligned} a(u,p) &= 0 \quad \text{for all } u \in H^{k+1}(\Omega),\ p \in P_\ell, \\ a(p,u) &= 0 \quad \text{for all } p \in P_k,\ u \in H^{\ell+1}(\Omega)\,. \end{aligned}$$

Then there exists a $0 < C < \infty$ such that

$$|a(u,v)| \le C|u|_{k+1,2}|v|_{\ell+1,2} \quad \text{for all } u \in H^{k+1}(\Omega),\ v \in H^{\ell+1}(\Omega)\,.$$

Here $|\cdot|_{r,2}$ is the order r Sobolev seminorm:

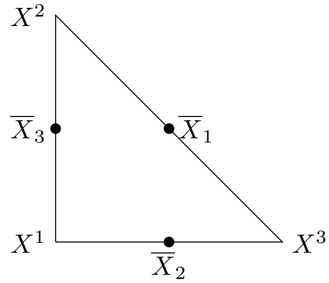
$$|v|_{r,2} = \left(\sum_{|\boldsymbol{\alpha}|=r} \|D^{\boldsymbol{\alpha}} v\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}.$$

Using the proof of the Bramble–Hilbert Lemma (see, e.g. [1, Lemma 4.25, pg. 224]) prove the result above.

Hint: This proof follows the same structure as the proof of the Bramble–Hilbert Lemma for sublinear functionals. The Poincaré inequality (Lemma 2.2.1 in the notes) may be useful.

[1] C. Grossmann, H-G. Roos, M. Stynes. *Numerical Treatment of Partial Differential Equations*. Universitext, Berlin Heidelberg, Germany, 2007.

Problem 6.3: Crouzeix–Raviart Element



The Crouzeix–Raviart element the triangular element with node functionals

$$\mathcal{N}_i(f) = f(\overline{X}_i) \quad \text{for } i = 1, 2, 3,$$

where $\overline{X}_i$ is the midpoint of the edge oppposite node i:

$$\overline{X}_i = \frac{1}{2} \sum_{j \neq i} X_j \,,$$

and the corresponding finite element space consists of piecewise linear functions.

(a) Show that this element is unisolvent.

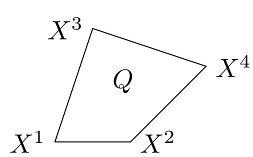
(b) For a mesh $\mathbb{T}$ consisting of Crouzeix–Raviart elements, is the resulting space $V_{\mathbb{T}} \subset C(\overline{\Omega})$?

Problem 6.4: Transformation of Quadrilaterals (For those who enjoy mindless computation...)

Transformation from the reference square $\widehat{Q} = [0,1]^2$ to a general quadrilateral given by vertices $X^i = (x_i, y_i)^T$ for $i = 1, \ldots, 4$, can be obtained by the mapping F given by

$$F(\boldsymbol{\xi}) = X^1(1-\xi)(1-\eta) + X^2\xi(1-\eta) + X^3(1-\xi)\eta + X^4\xi\eta.$$

Here, $\boldsymbol{\xi} = (\xi, \eta)^T$. The order of vertices follows the scheme



(a) Show that indeed $Q = F(\widehat{Q})$.

(b) Compute $\nabla F(\boldsymbol{\xi})$.

(c) Show that F is affine on each edge and independent of the vertices and independent of the vertices not adjacent to that edge.

(d) Compute the Jacobi determinant $J(\xi)$ and show that $J(\xi) \geq 0$, if and only if the quadrilateral is convex.

(e) Compute the eigenvalues of $(\nabla F(\boldsymbol{\xi}))^T \nabla F(\boldsymbol{\xi})$ and relate them to $\|F(\boldsymbol{\xi})\|$ and $\|(F(\boldsymbol{\xi}))^{-1}\|$.

(f) What happens to $J(\xi)$, $\|F(\boldsymbol{\xi})\|$ and $\|(F(\boldsymbol{\xi}))^{-1}\|$ at X^1 if X^2 gets close to X^1?