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Homework No. 5
Finite Elements, Winter 2018/19

Problem 5.1: Galerkin equations

Consider the one-dimensional problem

$$-u'' + u = f \quad \text{in } \Omega = (0,1),$$

for the space $V = H_0^1(\Omega)$.

We define the equidistant mesh

$$x_j = jh, \quad j = 0, \dots, n, n+1 \quad \text{with } h = \frac{1}{n+1}$$

on the interval Ω with $n+1$ mesh cells $T_j = (x_{j-1}, x_j)$. The finite-dimensional subspace V_n is now the span of piecewise linear functions

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & \text{if } x \in (x_{j-1}, x_j], \\ \frac{x_{j+1} - x}{h}, & \text{if } x \in (x_j, x_{j+1}), \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for } j = 1, 2, \dots, n.$$

(a) Sketch the domain Ω with its subdivision and a reasonable number of functions φ_j.

(b) Argue that the space V_n contains exactly all functions which are linear on each cell T_j, continuous on Ω and have zero boundary conditions.

(c) Set up the Galerkin equations.

(d) Compute the 2×2 cell matrices A_k and cell vectors $\boldsymbol{b}_k$.

(e) For $n = 4$, assemble the cell matrices and cell vectors into the global matrix A and the global vector $\boldsymbol{b}$. What does the matrix look like for general n?

(f) Calculate the solution of Galerkin equations for the right hand side $f(x) = 1$ and plot it for $n = 4$.

Problem 5.2: Integral node functionals

In two dimensions, a finite element on a triangle shall consist of the space of quadratic polynomials P_2, and shall utilise the node functionals $\mathcal{N}_i$ defined by

$$\mathcal{N}_i(f) = f(X^i) \qquad i = 1, 2, 3,$$
$$\mathcal{N}_i(f) = \frac{1}{|E_{i-3}|} \int_{E_{i-3}} f(x)\, \mathrm{d}s, \qquad i = 4, 5, 6.$$

Here, E_i is the edge of the triangle facing the vertex X^i, and $|E_i|$ is its measure.

(a) Show that this element is unisolvent.

(b) Derive a basis $\{\varphi_j\}$ for P_2 such that $\mathcal{N}_i(\varphi_j) = \delta_{ij}$.