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Homework No. 4
Finite Elements, Winter 2018/19

Problem 4.1: Regularity

Let $\Omega \subset \mathbb{R}^d$ be bounded, and consider the following problem

$$\begin{aligned} -\Delta u(x) &= f(x), && \text{in } \Omega, \\ u(x) &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Characterise the interior and global regularity of the solution u for the following scenarios.

(a) $\Omega = (-1, 1)$ and $f = \sin(x)$.

(b) $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < 1\}$ and $f = |\boldsymbol{x}|$.

(c) $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < 1\}$ and $f = \frac{1}{|\boldsymbol{x}|}$.

(d) $\Omega = \{\boldsymbol{x} = (r \sin\vartheta, r\cos\vartheta) \in \mathbb{R}^2 : 0 \le r < 1 \text{ and } \pi/4 < \vartheta < 2\pi\}$ and $f = |\boldsymbol{x}| = r$.

Problem 4.2: Elastic plates

"Plates" are flat bodies with (constant) positive, but small thickness. In linear Kirchhoff plate theory the vertical displacement of a plate can be described as solution u of the following fourth order pde

$$\Delta^2 u = f, \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and Δ^2 is the **biharmonic operator** defined by

$$\Delta^2 u(x, y) = \Delta(\Delta u(x, y)) = \partial_x^4 u(x, y) + 2\partial_x^2\partial_y^2 u(x, y) + \partial_y^4 u(x, y)$$

(a) Derive a symmetric weak formulation of the plate equation using integration by parts. Impose boundary conditions so that no boundary integrals appear.

(b) Prove the existence and uniqueness of a solution for $\Omega = (0, 1)^2$.
Hint: Think of a useful variant of the Poincaré inequality.

Problem 4.3: Fourier series approximation

Let $\Omega = (-1, 1)$ and consider the Fourier sine functions

$$\sin(k\pi x)\,, \quad \text{for } k = 1, 2, 3, \ldots$$

Show the following results by using the fact that the sine functions form a basis for $H_0^1(\Omega)$, which is orthonormal with respect to the $L^2(\Omega)$ inner product. That is, there exists $\alpha_k \in \mathbb{R}$ such that

$$f(x) \;=\; \sum_{k=1}^{\infty} \alpha_k \sin(k\pi x)\,, \quad \text{for } f \in H_0^1(\Omega), \tag{4.1}$$

and

$$\int_{-1}^{1} \sin(k\pi x)\sin(\ell\pi x)\;\mathrm{d}x \;=\; \begin{cases} 1 & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell\,. \end{cases}$$

(a) Prove Parseval's identity:

$$\|f\|_{L^2}^2 \;=\; \sum_{k=1}^{\infty} \alpha_k^2\,.$$

(b) For f given by (4.1), deduce a necessary condition on the geometric decay of $|\alpha_k|$ such that $f \in L^2(\Omega)$. Also, deduce a similar necessary condition on the geometric decay of $|\alpha_k|$ such that $f \in H_0^1(\Omega)$.

(c) Consider approximating f by the partial sum

$$f_n \;=\; \sum_{k=1}^{n} \alpha_k \sin(k\pi x)\,.$$

Determine the error in approximating f by f_n in $L^2(\Omega)$, and in $H_0^1(\Omega)$.

Hint: The following inequality may prove useful. For $s > 1$ and $n \in \mathbb{N}$ sufficiently large, one has

$$\sum_{k=n+1}^{\infty} \frac{1}{k^s} < \int_n^{\infty} t^{-s}\;\mathrm{d}t\,.$$