IWR – Universität Heidelberg
Prof. Dr. Guido Kanschat
Alec Gilbert

Due date: **2.11.2018**

Homework No. 3
Finite Elements, Winter 2018/19

Problem 3.1: Energy Norm

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $w \in L^\infty(\Omega)$ be such that $w(x) \geq w_0 > 0$ for all $x \in \Omega$.

(a) Show that

$$a(u, v) := \int_\Omega w(x) \nabla u(x) \cdot \nabla v(x) \; \mathrm{d}x, \qquad \|u\|_a := \sqrt{a(u, u)},$$

define a scalar product and a norm for the space $H_0^1(\Omega)$.

(b) Are they a scalar product and norm on the space $H^1(\Omega)$?

Problem 3.2: H^1 Regularity

Consider the domain $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$. Determine whether the following functions belong to $H^1(\Omega)$.

(a)

$$u(x) = \sin\left(\ln\left(\frac{1}{|x|}\right)\right)$$

(b)

$$u(x) = \frac{x}{|x|}$$

Problem 3.3: Weak formulation of Robin boundary value problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega$, and let $\mu > 0$. Consider the following Robin-boundary problem

$$\begin{aligned} -\Delta u(x) &= f(x), && \text{in } \Omega, \\ \partial_n u(x) + \mu u(x) &= g(x), && \text{on } \partial\Omega, \end{aligned}$$

(a) Formulate the problem weakly for functions $u \in H^1(\Omega)$.

(b) Equip $H^1(\Omega)$ with an inner product and a norm.

(c) Prove the existence and uniqueness of a solution to your weak formulation. What conditions did you assume that f and g satisfy?

(d) Set $\mu = 0$. Is there still a unique solution?

Problem 3.4: Bounded/Continuous Linear Operators

Let V, W be normed vector spaces, and let $A : V \to W$ be a linear operator. Prove that the following three statements are equivalent.

i. A is continuous at 0.

ii. A is continuous for all $v \in V$.

iii. A is bounded: there exists a constant $0 < C < \infty$ such that

$$\|Av\|_W \leq C \|v\|_V \quad \text{for all } v \in V\,.$$