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Homework No. 2
Finite Elements, Winter 2018/19

Problem 2.1: Given the sequence of functions

$$f_n(x) = \frac{|x|^3}{|x^2| + \frac{1}{n}}.$$

(a) Show that f_n is continuously differentiable.

(b) Show that $f_n \to |x|$ in $H^1(-1,1)$ as $n \to \infty$, where we equip $H^1(-1,1)$ with the norm

$$\|v\|_{H^1(-1,1)} = \left(\|v\|^2_{L^2(-1,1)} + \|v'\|^2_{L^2(-1,1)}\right)^{\frac{1}{2}} = \left(\int_{-1}^{1} \left(|v(x)|^2 + |v'(x)|^2\right) \mathrm{d}x\right)^{\frac{1}{2}}.$$

Hint: Use de l'Hôpital's rule for quotients of sequences which diverge to infinity.

Problem 2.2: Let $\Omega = (-1,1)$. Show that on the space of continuous functions on Ω the norms

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)| \quad \text{and} \quad \|f\|_2 = \left(\int_\Omega |f(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

are not equivalent.

Hint: Find a sequence which is bounded in one norm and tends to zero with respect to the other.

Problem 2.3: Trilinearform

For $\Omega \subset \mathbb{R}^2$ consider the term

$$c(\boldsymbol{w}; u, v) = (\boldsymbol{w} \cdot \nabla u, v), \quad c(\cdot;\cdot,\cdot) : W \times V \times V \to \mathbb{R}, \qquad \text{with } W = H^1_0(\Omega;\mathbb{R}^2), \quad \text{and} \quad V = H^1_0(\Omega).$$

Here $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product, and we equip the spaces V and W with the norms (2.3) and (2.4), respectively.

Note: The semicolon indicates that we will later on use $\boldsymbol{w}$ as data, such that $c(\boldsymbol{w};.,.)$ is a bilinearform which fits into our existing framework.

(a) Show that $c(\cdot;\cdot,\cdot)$ is linear in each variable (altogether trilinear) and continuous.

Hint: Show that $|c(\boldsymbol{w}; u, v)|$ is bounded. Use the Sobolev embedding theorem.

(b) Show that the following identity holds:

$$(\boldsymbol{w} \cdot \nabla u, u) = -\frac{1}{2}\left(\nabla\cdot\boldsymbol{w}, u^2\right).$$

Hint: The notation

$$\nabla\cdot\varphi := \frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2}$$

denotes the divergence of a sufficiently regular vector field $\varphi(x) = \left(\varphi_1(x), \varphi_2(x)\right)^T$.

(c) Deduce an analogous formula for $\boldsymbol{w} \in H^1(\Omega;\mathbb{R}^2)$ and $u \in H^1(\Omega)$ without zero boundary conditions.

Problem 2.4: Stationary convection-diffusion equation

Consider the stationary convection-diffusion equation

$$\begin{aligned} -\Delta u + \boldsymbol{w} \cdot \nabla u &= f, \qquad && \text{in } \Omega, \\ u &= 0, \qquad && \text{on } \partial\Omega \end{aligned}$$

for a bounded domain $\Omega \subset \mathbb{R}^2$ and a given function $\boldsymbol{w}\colon \Omega \to \mathbb{R}^2$ with $\nabla \cdot \boldsymbol{w} = 0$. Consider also the space $H_0^1(\Omega)$, which we equip with the Sobolev norm (2.3).

(a) Formulate the problem weakly for functions $u \in H_0^1(\Omega)$.

(b) Show that there is a unique solution of the weak formulation by using the theorem of Lax-Milgram. Make suitable assumptions on the data $\boldsymbol{w}$ and f. Why don't we use Riesz' representation theorem directly to show existence?

(c) What happens, if we do not assume $\nabla \cdot \boldsymbol{w} = 0$? Can we still guarantee existence?

Definition of norms

In general, let $\Omega \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$ be bounded, and denote $\boldsymbol{x} \in \Omega$ by $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$. We define the following norms,

$$\|v\|_{L^2(\Omega)} := \left(\int_\Omega |v(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}}, \qquad \text{for } v \in L^2(\Omega), \tag{2.1}$$

$$\|v\|_{H^1(\Omega)} := \left(\|v\|_{L^2(\Omega)}^2 + \sum_{j=1}^{d} \left\| \frac{\partial v}{\partial x_j} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \qquad \text{for } v \in H^1(\Omega), \tag{2.2}$$

$$\|v\|_{H_0^1(\Omega)} := \left(\sum_{j=1}^{d} \left\| \frac{\partial v}{\partial x_j} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \qquad \text{for } v \in H_0^1(\Omega). \tag{2.3}$$

For vector-valued functions $\boldsymbol{v} : \Omega \to \mathbb{R}^s$ (denoted $\boldsymbol{v} = (v_1, v_2, \ldots, v_s)$) we define the norm on $H_0^1(\Omega; \mathbb{R}^s)$ by

$$\|\boldsymbol{v}\|_{H_0^1(\Omega;\mathbb{R}^s)} := \left(\sum_{i=1}^{s} \|v_i\|_{H_0^1(\Omega)}^2 \right)^{\frac{1}{2}}, \qquad \text{for } \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^s). \tag{2.4}$$