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Revision No. 1
Finite Elements, Winter 2018/19

Problem 1.1: Chapter 1

(a) Consider a function $u \in L^2(\Omega)$. How can you formulate the weak Laplacian for this function?

(b) Formulate the Riesz representation theorem and the Lax-Milgram lemma.

(c) In which sense is the Lax-Milgram lemma an enhancement of the Riesz representation theorem?

(d) What does V-elliptic mean for a bilinear form $a(u, v)$ with u, v in V?

(e) Is the bilinear form $a(u, v) = (\nabla u, \nabla v)$ V-elliptic for $V = H^1(\Omega)$?

(f) Given an elliptic PDE defined on a convex domain, what is the highest possible smoothness that the solution can have? Describe the other conditions required to obtain this smoothness.

Problem 1.2: Chapter 2

(a) Explain what is meant by the term 'Galerkin orthogonality'.

(b) Formulate Ceá's lemma and state the requirements for this result.

(c) Formulate a linear error functional $J(\varphi)$ which represents the L^2-Norm for $\varphi = u - u_h$.

(d) Describe the duality argument (Aubin-Nitsche trick) for error estimates in 'weak' norms. What is it used for?

(e) How do node values induce continuity into a finite element space?

(f) What does the term unisolvence mean for a polynomial ansatz space?

(g) When is a triangulation called shape regular?

(h) Write a typical a priori error estimate.

(i) State the Bramble-Hilbert Lemma.

(j) Describe the connection between error estimates, transformation of the reference cell and the Bramble–Hilbert Lemma.

(k) Which order of convergence can we obtain for the L^2- and the H^1-norm of an approximation with quadratic finite elements in the best case?

(l) What is the difference between a-priori and a-posteriori error estimates?

(m) Describe the concept of a dual weighted error estimator.

Problem 1.3: Chapter 3

(a) State Strang's first lemma. What are the consequences of this lemma for implementing a finite element method in practice?

(b) For a PDE with a general coefficient on the second order term, how would you construct the stiffness matrix in practice?

(c) For a quadratic finite element space, what order quadrature rule is required to maintain the optimal order of convergence for the finite element method?

Problem 1.4: Chapter 4

(a) How many steps does the conjugate gradient (CG) method, using exact arithmetic, take to converge to the true solution? Why is this property useless in practice?

(b) Of what order is the condition number of the mass matrix and the stiffness matrix?

(c) What effect does **(b)** have on solving a finite element problem in practice?

(d) What is a 'preconditioner'? What purpose does it serve?

Problem 1.5: Miscellaneous

(a) Describe the steps you would take to approximate the solution to Poisson's equation with Dirichlet boundary conditions on a bounded convex domain.

(b) Consider an elliptic PDE on a bounded domain with C^2 boundary, and with smooth coefficients and right hand side. Describe a FE space in which you would approximate the solution to this problem. How does the FE solution converge to the true solution?

(c) What are the advantages of considering the weak form of a PDE?