IWR – Universität Heidelberg
Prof. Dr. Guido Kanschat
Alec Gilbert

Due date: **19.10.2018**

# Homework No. 1
# Finite Elements, Winter 2018/19

## Problem 1.1: Variational equations in $\mathbb{R}^n$

Given a symmetric, positive definite matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$ and the "energy functional"

$$E(x) = \frac{1}{2}x^T Ax - x^T b, \tag{1.1}$$

**(a)** Derive the variational equation of the minimization problem by studying the derivative of the auxiliary function $\Phi(t) = E(x + ty)$ for arbitrary $y \in \mathbb{R}^n$.

**(b)** Show that a vector $x \in \mathbb{R}^n$ minimizes $E(x)$, that is,

$$E(x) \le E(y) \quad \forall y \in \mathbb{R}^n,$$

if and only if

$$Ax = b.$$

**(c)** Conclude that the minimizer $x$ exists and is unique.

## Problem 1.2: Minimizing sequence

**(a)** Using the knowledge from Problem 1.1 that a minimiser $x$ of the energy functional in (1.1) exists, show that a sequence $\{x^{(k)}\}$ that satisfies

$$E(x^{(k)}) \to \inf_{y\in\mathbb{R}^d} E(y), \tag{1.2}$$

necessarily converges to $x$. The "binomial formula" $x^T Ax - y^T Ay = (x + y)^T A(x - y)$ and the fact that $A$ is invertible are useful ingredients to this proof.

**(b)** Show without assuming the existence of the minimizer $x$, that a sequence $\{x^{(k)}\}$ for which (1.2) holds is necessarily a Cauchy sequence. Can you conclude the existence of a minimizer $x$?

## Problem 1.3: Integration by parts

Let $\Omega$ be a domain in $\mathbb{R}^d$. Use the Gauß theorem for smooth vector fields $\varphi : \Omega \to \mathbb{R}^d$, namely,

$$\int_\Omega \nabla\cdot\varphi \,\mathrm{d}x = \int_{\partial\Omega} \varphi \cdot \mathbf{n} \,\mathrm{d}s,$$

to show Green's first and second formula (for smooth scalar functions $u$ and $v$)

$$-\int_\Omega \Delta u v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} \partial_n u v \,\mathrm{d}s$$

$$\int_\Omega (u\Delta v - v\Delta u)\,\mathrm{d}x = \int_{\partial\Omega} (u\partial_n v - v\partial_n u)\,\mathrm{d}s.$$

Here, $\mathbf{n}$ is the outward unit normal vector to $\Omega$ on $\partial\Omega$. The differential operators have the meaning:

| | |
|---|---|
| $\nabla u = (\partial_1 u, \ldots, \partial_d u)^T$ | gradient |
| $\partial_n u = \mathbf{n} \cdot \nabla u$ | normal derivative |
| $\nabla\cdot\varphi = \partial_1\varphi_1 + \cdots + \partial_d\varphi_d$ | divergence |
| $\Delta u = \nabla\cdot\nabla u = \partial_{11}u + \cdots + \partial_{dd}u$ | Laplacian |

## Problem 1.4: Friedrichs' inequality

**(a)** Prove Friedrichs' inequality

$$\|u\|_{L^2(\Omega)} \le c\|u'\|_{L^2(\Omega)}, \qquad \text{with } c = b - a$$

for $\Omega = (a, b)$ and functions $u \in C^1_0(\Omega)$.

**(b)** Generalize the proof for functions in $H^1_0(\Omega)$, using that each function in $H^1_0(\Omega)$ is the limit of a sequence in $C^1_0(\Omega)$.